Broken harmonic functions

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October 17, 2014

Abstract

Revisiting our theory of functions of good variation (FGV) we introduce a family of functions allowing us to determine the good variation index of the Ingham function $\Phi(x) = x\left\lfloor \frac{1}{x} \right\rfloor$ [1] providing new tauberian conjectures.

Introduction

With our tauberian approach[2] we believe that the difficulty of RH relies on the fact that it is not totally an arithmetical problem nor totally an analytical problem. A subtle use of both complex analysis and arithmetic seems needed to settle the problem. Indeed although our functions of good variation are discretely defined (see section 1 for the discrete implicit definition) an analytic conjecture is crucial in order to get information on the good variation index of the Ingham function. In addition this analytic conjecture contains an important discrete condition: the Hardy-Littlewood-Ramanujan criteria (HLR criteria) wich appears to be the corner stone of our strategy. This strategy depends upon 2 conjectures.

In section 2 we indroduce the so called broken harmonic functions and state the comparison conjecture between good variation index of these functions.

In section 3 we consider a special case of broken harmonic functions involving $\sqrt{2}$ and state a conjecture on the value of its good variation index.

The section 4 described how these 2 conjectures allow us to derive RH is true.

Finally in section 5 we generalise the Ingham function and discuss the extension of the method to $L-$functions.

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[1] The name comes from a tauberian theorem of Ingham which was the starting point of our study which began 4 years ago. Namely

$$na_n \geq -C \wedge \lim_{n\to\infty} \sum_{k=1}^{n} a_k \Phi\left(\frac{k}{n}\right) = l \Rightarrow \sum_{k=1}^{\infty} a_k = l$$

Cf.[Kor], theorem 18.2.p.110 or [Ten], corollaire II.7.23.

[2] Our preliminary study is described in rough unpublished expository papers[Clo1, Clo2, Clo3].

This comparison method reveals somewhat a fractal (or multifractal) aspect of RH which is hidden behind the complexity of the Mertens function $M(n) = \sum_{k=1}^{n} \mu(k)$ or the Liouville function $L(n) = \sum_{k=1}^{n} \lambda(k)$. This fact is well illustrated by the figure 4 in section 4.

1 Functions of good variation (FGV)

1.1 Implicit discrete definition

Let $(a_n)_{n\geq 1}$ be a real sequence and $g : [0,1] \to \mathbb{R}$ be a bounded and measurable function. Defining the sums:

- $A(n) := \sum_{k=1}^{n} a_k$
- $A_g(n) := \sum_{k=1}^{n} a_k g\left(\frac{k}{n}\right)$

we say that g is a FGV of index $\alpha(g) \in \mathbb{R}$ if we have the 2 following tauberian conditions:

1. $(\beta < \alpha(g)) \ \wedge \ A_g(n) \sim n^{-\beta} \ \ (n \to \infty) \Rightarrow \lim_{n\to\infty} n^{\beta} A(n)$ exists.
2. $(\beta > \alpha(g)) \ \wedge \ A_g(n) \sim n^{-\beta} \ \ (n \to \infty) \Rightarrow n^{\beta} A(n)$ is unbounded

It is easy to see that FGV exist. For instance polynomials are FGV. More precisely letting $g(x) = \sum_{k=0}^{m} c_k x^k$ with $m \geq 1$, $c_m \neq 0$ and $g(0)g(1) \neq 0$ we can say that g is a FGV of index $\alpha(g) = \min\{\Re(\rho_i)\}$ where ρ_i are the roots of $\sum_{k=0}^{m} \frac{c_k}{k-x}$ (cf. [Clo3] for a sketch of proof).

1.2 Analytic conjecture

We propose an analytic method to determine the good variation index $\alpha(g)$ of some FGV g. This method doesn't work for all FGV. For instance the "almost Dirac" function defined by $g(x) = 1$ on $[0,1]$ except at $x = \frac{1}{2}$ where we have $g\left(\frac{1}{2}\right) = r \in]0,1[$ is a FGV of index $\alpha(g) = -\frac{\log(1-r)}{\log 2}$ which can be proved only by arithmetical means (cf. [Clo1] p. 12 and the sketch of proof) .

Before stating this conjecture let us define a function of quasi bounded variation on $]0,1]$, the HLR criteria and the little Mellin transform.

Function of quasi bounded variation

We say that a bounded and measurable function $g :]0,1] \to \mathbb{R}$ is of quasi bounded variation if we have:

1. $\forall x \in]0,1]$ g is of bounded variation on $[x,1]$.
2. $\exists x_0 \in]0,1]$ such that g is monotonic by parts on $]0,x_0]$.

HLR criteria (for Hardy-Littlewood-Ramanujan)[3]

We say that a bounded and measurable function g satisfies the HLR criteria if it is a FGV of index $0 \leq \alpha(g) \leq 1$ such that for any function $f(x)$ converging exponentially fast to zero as $x \to \infty$ one has

$$A_g(n) = f(n) \Rightarrow \forall \varepsilon > 0 \lim_{n \to \infty} a(n) n^{1-\varepsilon} = 0$$

Little Mellin transform

We introduce $g^\star$ as the little Mellin transform of a bounded and measurable function g as follows when $\Re z < 0$

$$g^\star(z) := \int_0^1 g(t) t^{-z-1} dt$$

We can now state the analytic conjecture.

Analytic conjecture

Let g be a function of quasi bounded variation $]0,1] \to \mathbb{R}$ satisfying $g(1) \neq 0$ and $\lim_{x\to 0} g(x) \neq 0$ which has the following properties:

- g is continuous on the left.
- g satisfies the HLR criteria.
- $g^\star$ can be extended analytically to the whole complex plane with possibly some singularities.
- $\inf \{\Re(\rho) \mid \rho \in \mathbb{C} \wedge g^\star(\rho) = 0\} \in [0,1]$.

Then g is a FGV of index $\alpha(g) = \inf \{\Re(\rho) \mid \rho \in \mathbb{C} \wedge g^\star(\rho) = 0\}$.

Remark

This conjecture is true for polynomials and for continuous functions of bounded variation which are well approximated by polynomials. Using results from tauberian theory related to the Mellin convolution ([BGT, Bin]), we think the conjecture is also true for any function of bounded variation on $[0,1]$. For functions of quasi bounded variation on $]0,1]$ the HLR criteria seems to play a central role.

[3] The name HLR comes from the condition $a(n) = O\left(\frac{1}{n}\right)$ in the first tauberian theorem of Hardy-Littlewood and from the Ramanujan conjecture related to the behaviour of the coefficients of Dirichlet series in the Selberg class (this remark will be relevant in 5.5.1.).

2 Comparison conjecture

At the early stage of our study on FGV we thought that comparing Φ to a simpler function could shed light on the value of $\alpha(\Phi)$. Unfortunately our ideas were too naive. Essentially we were confronted to the same problem than Turan with his approach to RH based on the location of zeros of sections of the zeta function[Bor1]. Indeed approaching the Ingham function by functions with a finite number of discontinuities [Clo1] led us to the same kind of difficulties. That's why we are convinced that functions with infinitely many discontinuities are needed and recently we came across a useful family of functions.

2.1 Broken harmonic functions

Let a, b be two reals satisfying $-1 \leq a < 1,\ 0 < a+b \leq 1$ and define x_n by $x_0 = 1,\ x_1 = \frac{b}{1-a}$ and for $n \geq 2$

$$x_{n+1} = \frac{bx_n}{1 - ax_n}$$

Then we define the broken harmonic function $g_{(a,b)}$ for $0 < x \leq 1$ as follows

$$x_{n+1} < x \leq x_n \Rightarrow g_{(a,b)}(x) = \frac{x}{x_n}$$

Examples

- $g_{(-1,1)} = \Phi$ which has its discontinuities at $x = \frac{1}{k}$ for $k \geq 2$ integer.
- $g_{(-1/2,1)}(x) = \frac{x}{2} \left\lfloor \frac{2}{x} \right\rfloor$.
- For $r > 1$ we have $g_{(0,1/r)}(x) = xr^{\left\lfloor -\frac{\log x}{\log r} \right\rfloor}$.

As we shall see the functions $g_{(0,1/r)}$ will be very interesting for our purpose. In the sequel BHF will stand for broken harmonic function.

2.2 Comparison conjecture

BHF deserve attention since they are all built in a same way and their good variation index seem to follow rules. Hence we elaborated a comparison conjecture between them. The strong form of the conjecture relates all BHF whereas the weak form is restricted to the Ingham function and to $g_{(0,1/r)}$ when $r > 1$.

First of all we conjecture that BHF are FGV which can be devided in 2 sets. The set of BHF which satisfy the HLR crieria and the set of BHF which don't satisfy the HLR criteria.

Strong form of the comparison conjecture

Let a, b, a', b' be 4 real values verifying:

- $-1 \leq a < 1$, $0 < a+b \leq 1$, $-1 \leq a' < 1$, $0 < a'+b' \leq 1$.

Suppose $g_{(a,b)}$ satisfies the HLR criteria and $g_{(a',b')}$ doesn't satisfy the HLR crieria. Then we have the following inequality between index of good variation

$$\alpha\left(g_{(a,b)}\right) \geq \alpha\left(g_{(a',b')}\right)$$

Weak form of the comparison conjecture

Let $r > 1$. If $g_{(0,1/r)}$ doesn't satisfy the HLR criteria then we have the following inequality between its index and the index of Φ which satisfies the HLR criteria

$$\alpha(\Phi) \geq \alpha\left(g_{(0,1/r)}\right)$$

Remark There is a rough heuristic argument supporting the conjecture. Let us define 2 sequences a and a' as follows

$$\sum_{k=1}^{n} a_k g_{(a,b)}\left(\frac{k}{n}\right) = \sum_{k=1}^{n} a'_k g_{(a',b')}\left(\frac{k}{n}\right) = \frac{1}{2^n}$$

then we have according to FGV properties:

- $\sum_{k=1}^{n} a_k \ll n^{-\alpha\left(g_{(a,b)}\right)+\varepsilon}$
- $\sum_{k=1}^{n} a'_k \ll n^{-\alpha\left(g_{(a',b')}\right)+\varepsilon}$

Now if $g_{(a,b)}$ satisfies the HLR criteria but not $g_{(a',b')}$ then we have some chances to get for N large enough

$$\max\left\{\left|\sum_{k=1}^{n} a_k\right| \quad 1 \leq n \leq N\right\} < \max\left\{\left|\sum_{k=1}^{n} a'_k\right| \quad 1 \leq n \leq N\right\}$$

because a'_n is bigger in absolute value than a_n.

This would mean $\alpha\left(g_{(a,b)}\right) \geq \alpha\left(g_{(a',b')}\right)$. In section 4 we provide an example using the Liouville function (fig. 4) where this phenomenom seems true.

3 On the good variation index of $g_{(0,1/\sqrt{2})}$

It is this function which will show the relevance of the comparison conjecture. Firstly the analytic conjecture does'nt allow us to determine the good variation index of $g_{(0,1/\sqrt{2})}$. Indeed we have

$$g^{\star}_{(0,1/r)}(z) = \frac{1-r^{1-z}}{(1-z)(1-r^z)}$$

Therefore when $g_{(0,1/r)}$ satisfies the HLR criteria we have simply $\alpha\left(g_{(0,1/r)}\right) = 1$ which is the case when $r \geq 2$ is an integer value (and we believe it is the only case).

When r isn't an integer value however $g_{(0,1/r)}$ clearly doesn't satisfy the HLR criteria and experiments show that $\alpha\left(g_{(0,1/r)}\right) = 0$ very often and sometime the index is not trivial i.e. $0 < \alpha\left(g_{(0,1/r)}\right) < 1$.

The simplest case for which the index seems not trivial is $r = \sqrt{2}$ since we conjecture that we have

$$\alpha\left(g_{(0,1/\sqrt{2})}\right) = \frac{1}{2}$$

Experimental support

Let $a(n)$ be defined by the recursion

$$A_{g_{(0,1/\sqrt{2})}}(n) = \frac{1}{2^n}$$

Then we plot $A(n)\sqrt{n}$ (fig. 1) where a fractal pattern appears which seems to stay bounded along the y axis and just after we plot $a(n)\sqrt{n}$ (fig. 2) showing clearly that $g_{(0,1/\sqrt{2})}$ doesn't satisfy the HLR criteria.

fig.1

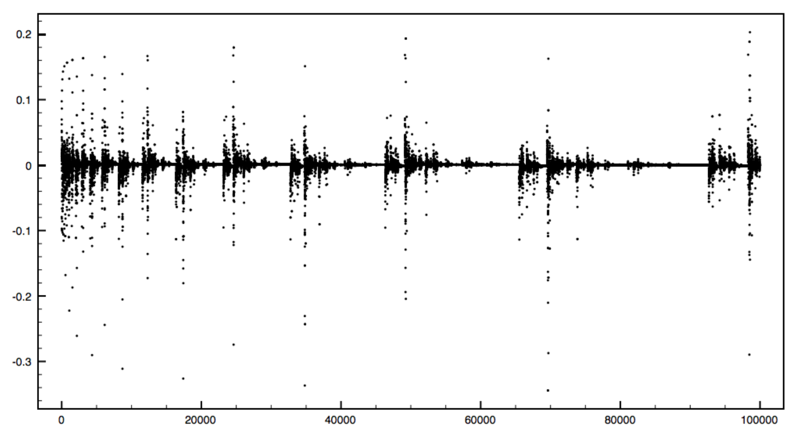
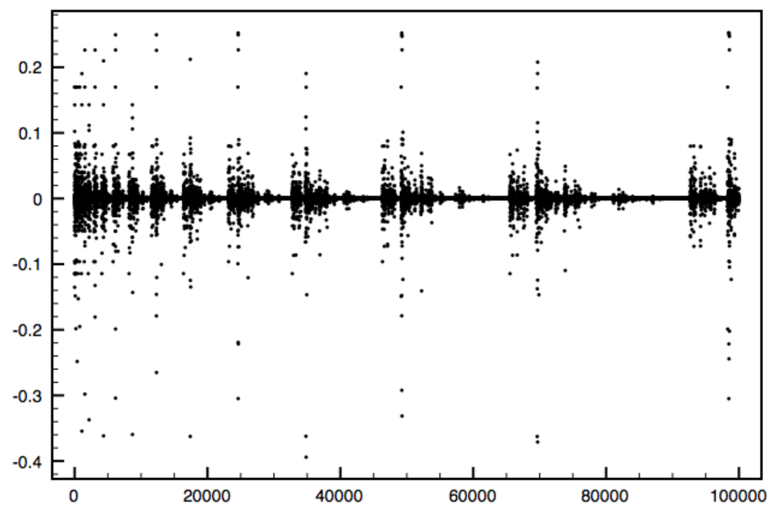
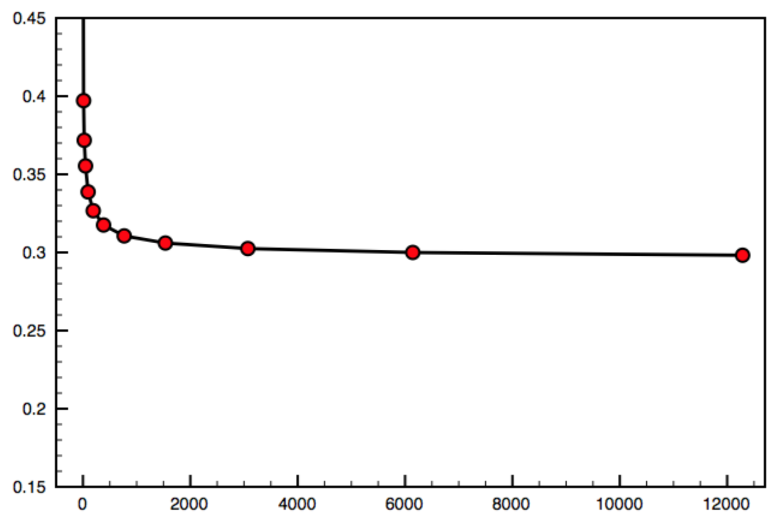


fig.2



Hereafter we compute $a(n)$ defined by $A_{g_{(0,1/\sqrt{2})}}(n) = \frac{1}{n}$. We can notice that local maxima of $A(n)$ in the halfplane $y > 0$ occur at values of n of form 3.2^k. Therefore we plot $A(3.2^k)\sqrt{3.2^k}$ for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$.

fig.3



The graph could converge or stay bounded by a slowly varying function supporting the conjecture $\alpha\left(g_{(0,1/\sqrt{2})}\right)=\frac{1}{2}$.

Remark

$\alpha\left(g_{(0,1/\sqrt{2})}\right)=\frac{1}{2}$ is an interesting conjecture in its own and should be confirmed by more experiments. In general we think that:

- $r\geq 2$ integer yields $\alpha\left(g_{(0,1/r)}\right)=1$ (see APPENDIX 2 for some examples).
- $r=m^{1/k}$ for $k\geq 2$ fixed integer and $m\geq 2$ integer but ot a power of k yields $\alpha\left(g_{(0,m^{-1/k})}\right)=\frac{1}{k}$ (see APPENDIX 3 where we take $r=2^{\frac{1}{3}}$).
- If r^j is never an integer value for $j\geq 1$ integer then $\alpha\left(g_{(0,1/r)}\right)=0$ (see APPENDIX 4 for examples).

Right now we have no clue to show these assertions. However the fractal aspect of graphics suggest that a proof could exist. Some diophantin facts like $\left|2^{-N-\frac{1}{2}}-\frac{k}{n}\right|>\frac{C(N)}{n^2}$ should also play a role.

4 On RH

We have $\Phi^\star(z)=\frac{\zeta(1-z)}{1-z}$ and Φ satifies the HLR criteria and the other conditions of the analytic cojecture. Hence we have thanks to the known location of the zeros of the zeta function

$$0<\alpha\left(\Phi\right)\leq\frac{1}{2}\tag{1}$$

next since $g_{(0,1/\sqrt{2})}$ doesn't satisfy the HLR criteria from section 3 (fig.2) the comparison conjecture (strong form or weak form) and the conjecture $\alpha\left(g_{(0,1/\sqrt{2})}\right)=\frac{1}{2}$ yield

$$\alpha\left(\Phi\right)\geq\alpha\left(g_{(0,1/\sqrt{2})}\right)=\frac{1}{2}\tag{2}$$

whence from (1) and (2) we get

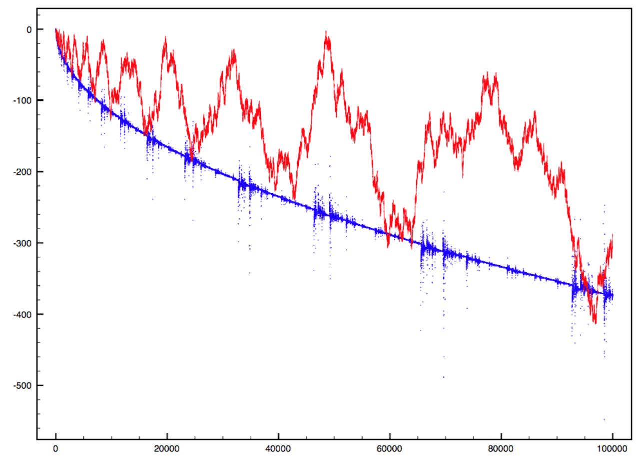
$$\alpha\left(\Phi\right)=\frac{1}{2}$$

and $\zeta(1-z)$ has no zero in the half-plane $\Re z<\frac{1}{2}$.

Exeprimental comparison involving the Liouville function

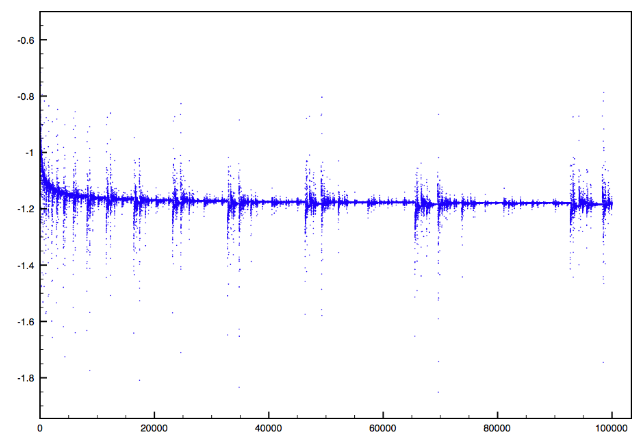
Comparing things at the limit of the index is interesting. Hence let $A_{g(0,1/\sqrt{2})}(n) = \frac{1}{\sqrt{n}}$ we plot $\sum_{k=1}^{n}(-1)^{\Omega(k)}$ (red) vs $\sum_{k=1}^{n} ka(k)$ (blue)

fig.4



Plot of $n^{-1/2}\sum_{k=1}^{n} ka(k)$

fig.5



The plot behaves like a very slowly varying function (of type $\log(n)^{\varepsilon}$) and the fig.4 shows that the red graphic doesn't come across the minima of the blue graphic. This would mean $\sum_{k=1}^{n}(-1)^{\Omega(k)} \ll n^{1/2+\varepsilon}$ implying RH is true.

5 Generalisation

5.1 Generalisation of the Ingham function

Let χ be a caracter and let us define the function g_χ on $]0,1]$

$$g_\chi(x) = x \sum_{1 \leq k \leq 1/x} \chi(k) \left\lfloor \frac{1}{kx} \right\rfloor$$

Then we have

- g_χ satisfies the HLR criteria
- $g_\chi^\star(z) = \frac{\zeta(1-z)L(1-z,\chi)}{1-z}$

Thus we can extend the comparison conjecture providing a broader definition of BHF.

5.2 General definition of broken harmonic functions

This definition is an extension of the construction in section 2. Let u be a real sequence satisfying

- $1 = u_1 > u_2 > u_3 > ... > u_\infty = 0$.

Let g_u be the left continuous function defined for $n \geq 1$ and any $x \in]0,1]$ by

$$u_{n+1} < x \leq u_n \Rightarrow g_u(x) = p_n x$$

where $p_n > 0$ is an increasing real sequence such that $u_i p_i \ll 1$. Then g_u is a BHF.

5.3 Generalisation of the comparison conjecture

If g_u is a BHF satisfying the HLR criteria and $g_{u'}$ is a BHF which doesn't satisfy the HLR criteria then we have

$$\alpha\left(g_u\right) \geq \alpha\left(g_{u'}\right)$$

5.4 Application to the generalised Riemann hypothesis

RH is true for $L(s,\chi)$.

Proof

Without loss of generality we take the caracter modulo 4:

- $\chi = 1, 0, -1, 0, 1, 0, -1, 0, ...$

giving $L(s,\chi) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^s}$ the Dirichlet beta function. Let $u_i = \frac{1}{v_i}$ where v is the increasing sequence of integers which can be written as a sum of 2 squares. Then keeping the notations in 5.2 and letting $p_i = v_i g_\chi\left(\frac{1}{v_i}\right)$ we get

$$g_\chi = g_u$$

which is a BHF. Then from 5.1 and the analytic conjecture we have from the known location of the zeros of $\zeta(1-s)L(1-s,\chi)$

$$0 < \alpha\left(g_\chi\right) \leq \frac{1}{2} \tag{3}$$

Next using again $\alpha\left(g_{(0,1/\sqrt{2})}\right) = \frac{1}{2}$ and the fact that $g_{(0,1/\sqrt{2})}$ is a BHF which doesn't satisfy the HLR criteria we get from the conjecture 5.3

$$\alpha\left(g_\chi\right) \geq \frac{1}{2} \tag{4}$$

Finally (3) and (4) yield $\alpha\left(g_\chi\right) = \frac{1}{2}$ and $\zeta(1-s)L(1-s,\chi)$ has no zero in the half-plane $\Re s < \frac{1}{2}$.

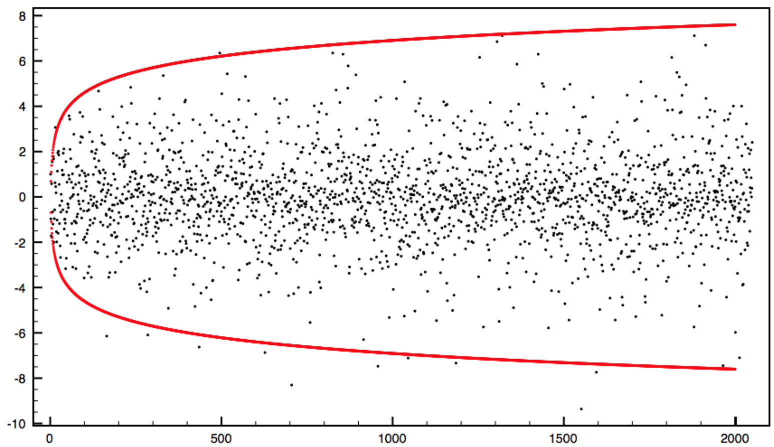
5.5 Remarks

5.5.1 Extension to other L functions

The method in section 5 for Dirichlet L functions can be extended naturally to other L functions. For instance let us consider the Ramanujan τ function and the associated automorphic form. It suffices to take the following function

$$g_\tau(x) = x \sum_{1\leq k\leq 1/x} \frac{\tau_k}{k^{11/2}} \left\lfloor \frac{1}{kx} \right\rfloor$$

which is a BHF and we believe that it satisfies the HLR criteria. Indeed computing 2000 terms of $a(n)$ given by the recursion $A_{g_\tau}(n) = \frac{1}{2^n}$ it seems that g_τ satisfies the HLR criteria as shown by the graphic below (fig. 6) where we compare $na(n)$ to $\pm \log n$ (red) fig. 6



It looks like $a(n)n$ stays of order $\log n$ thus we would have $\lim_{n\to\infty} a(n)n^{1-\varepsilon} = 0$ for any $\varepsilon > 0$ and g_τ would satisfy the HLR criteria.

5.5.2 On the zeta function of Davenport and Heilbronn

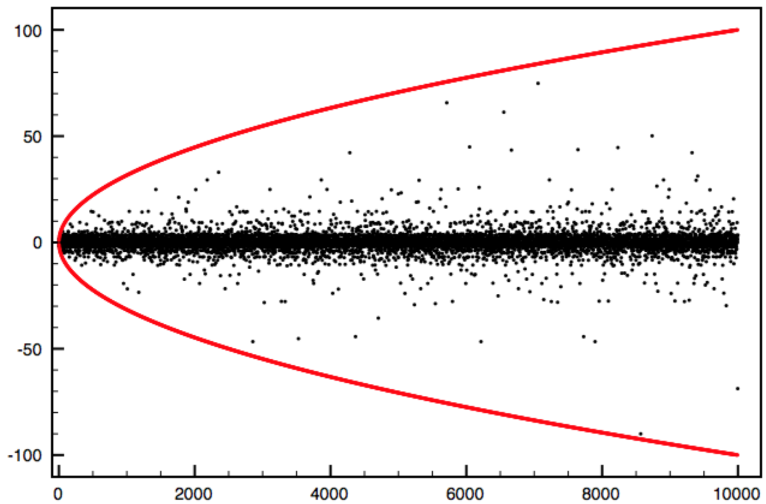
It is interesting to see that our method doesn't work for this function.

We consider $\zeta_H(s) = \sum_{n\geq 1} \frac{h(n)}{n^s}$ where h is the 5-periodic sequence $[1, \xi, -\xi, -1, 0]$ where $\xi = \frac{-2+\sqrt{10-2\sqrt{5}}}{-1+\sqrt{5}}$. It is known that ζ_H satisfies a functional equation like ζ and has nontrivial zeros off the critical line. Letting

$$g_H(x) = x \sum_{k\geq 1} h(k) \left\lfloor \frac{1}{kx} \right\rfloor$$

and computing 10000 terms of $a(n)$ given by the recursion $A_{g_H}(n) = \frac{1}{2^n}$ it is clear that g_H doesn't satisfy the HLR criteria as shown below (fig. 7) where we have compared $na(n)$ with $\pm\sqrt{n}$ (red).

fig. 7



$na(n)$ seems not bounded by a slowly varying function and we wouldn't have $\lim_{n\to\infty} a(n)n^{1-\varepsilon} = 0$ for any $\varepsilon > 0$. Therefore g_H doesn't satisfy the HLR criteria and we can say nothing about the value of $\alpha(g_H)$.

The fact that $h(n)$ is not multiplicative and that ζ_H has no Euler product is certainly the cause of this phenomenon. Indeed to us the HLR criteria has something to do with the quasi-multiplicity of a_n which is a concept that will be described in more details later.

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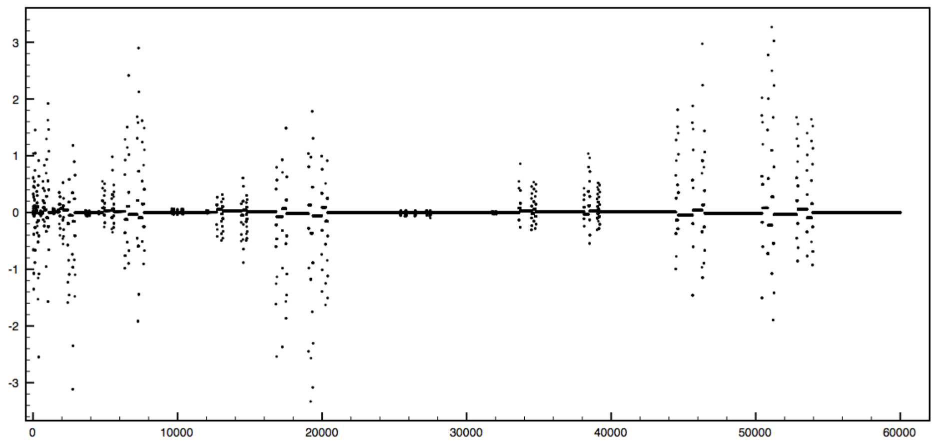
APPENDIX 1

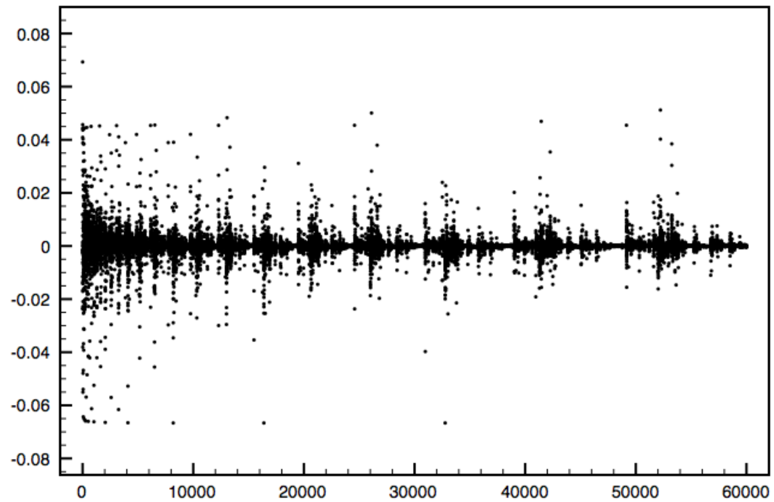
$A_{g_{(0,1/\sqrt{7})}}(n) = \frac{1}{n^2}$ plot of $A(n)\sqrt{n}$



It could be bounded and so $\alpha\left(g_{(0,1/\sqrt{7})}\right) = \frac{1}{2}$.

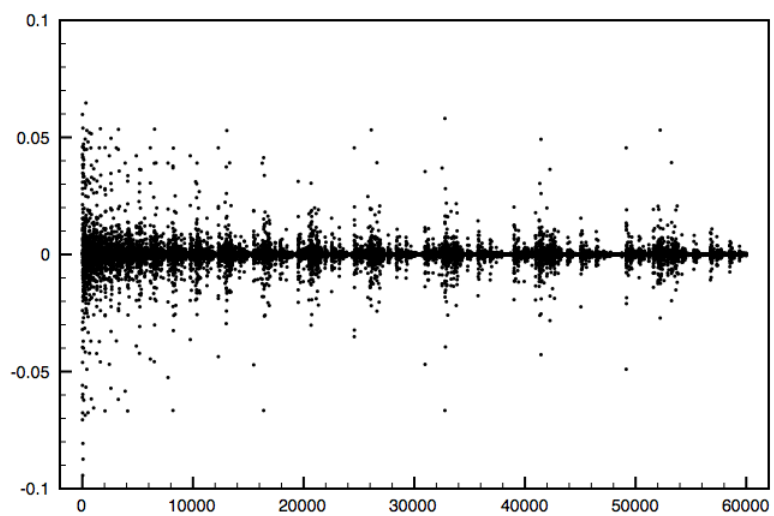
APPENDIX 2

$A_{g_{(0,2^{-1/3})}}(n) = \frac{1}{2^n}$ plot of $A(n)n^{1/3}$



It seems bounded and $\alpha\left(g_{(0,2^{-1/3})}\right) = \frac{1}{3}$ is probable.

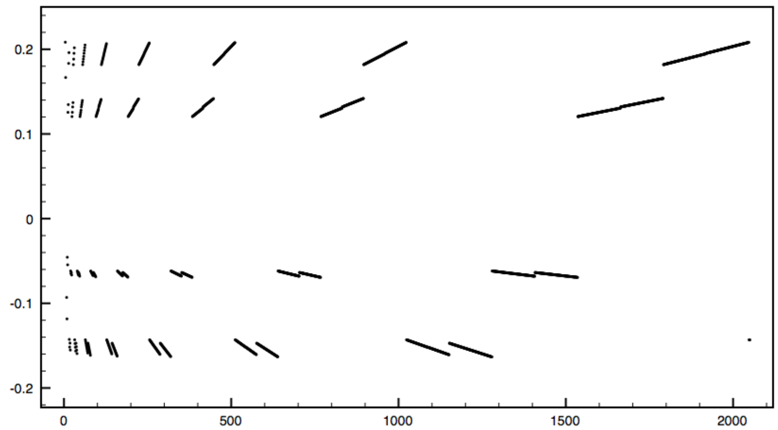
$A_{g_{(0,2^{-1/3})}}(n) = \frac{1}{2^n}$ plot of $a(n)n^{1/3}$



$g_{(0,2^{1/3})}$ doesn't satisfy the HLR criteria.

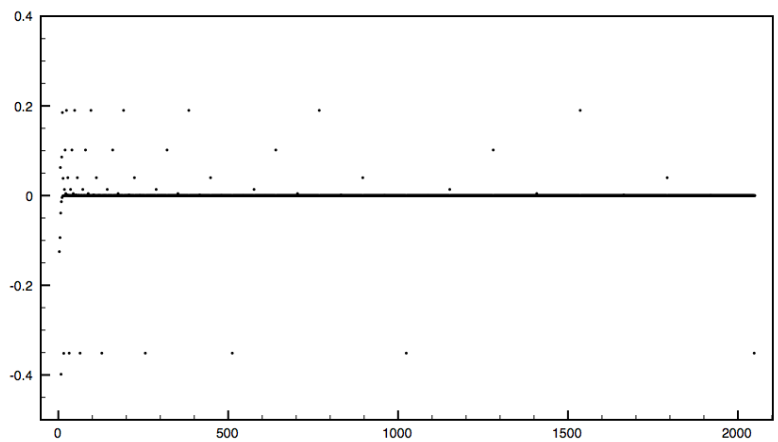
APPENDIX 3

$A_{g_{(0,1/2)}}(n) = \frac{1}{2^n}$ plot of $A(n)n$



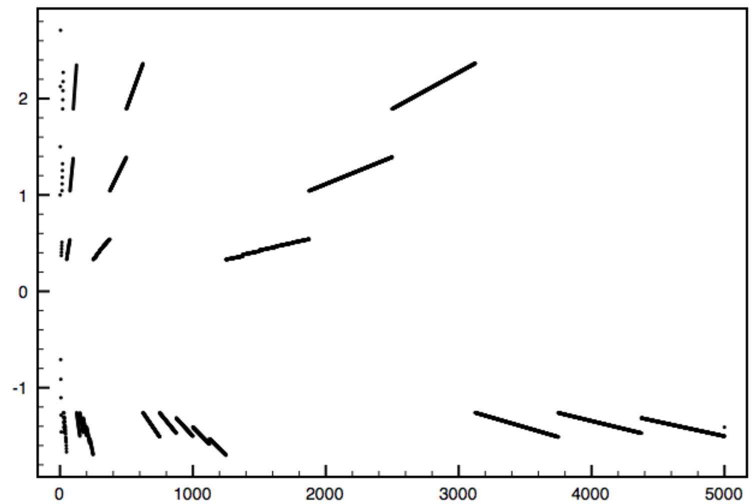
It is clearly bounded with a simple fractal pattern.

$A_{g_{(0,1/2)}}(n) = \frac{1}{2^n}$ plot of $a(n)n$

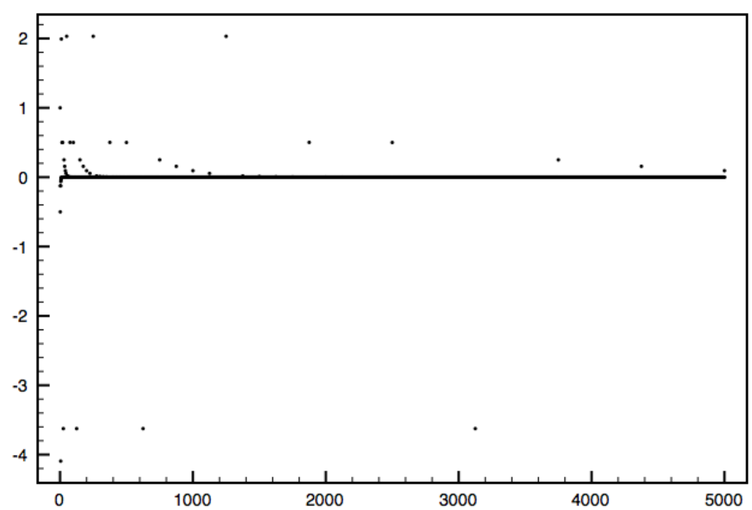


The HLR criteria is satisfied.

$A_{g_{(0,1/5)}}(n) = \frac{1}{2^n}$ plot of $A(n)n$

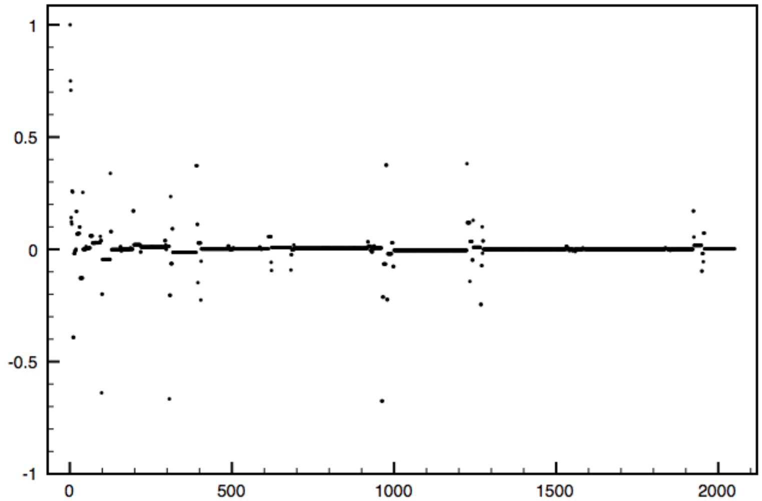


$A_{g_{(0,1/5)}}(n) = \frac{1}{2^n}$ plot of $a(n)n$



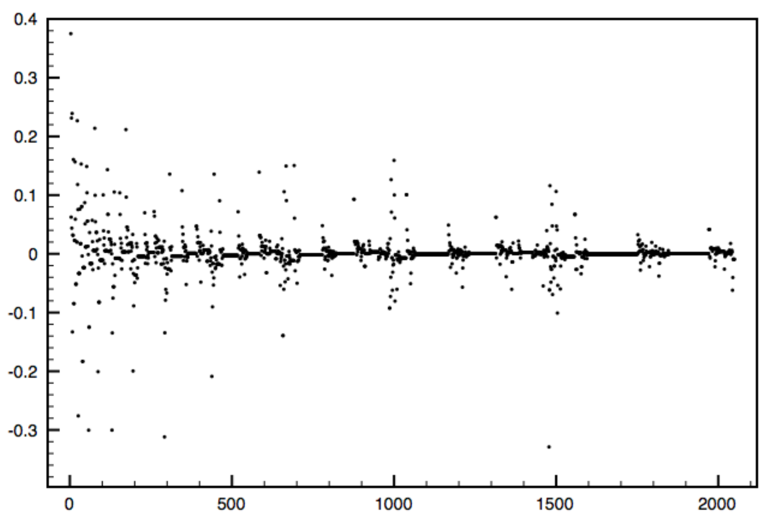
APPENDIX 4

$A_{g_{(0,\pi)}}(n) = \frac{1}{2^n}$ plot of $A(n)$



It seems bounded and so $\alpha\left(g_{(0,1/\pi)}\right) = 0$.

$A_{g_{(0,2/3)}}(n) = \frac{1}{2^n}$ plot of $A(n)$



It seems bounded and so $\alpha\left(g_{(0,2/3)}\right) = 0$.